Available online at www.sciencedirect.com
SCIENCE DIRECT
ELSEVIER
Journal of APPLIED MATHEMATICS AND MECHANICS
www.elsevier.com/locate/jappmathmech
Journal of Applied Mathematics and Mechanics 69 (2005) 553–563

THE ASYMPTOTIC STABILITY OF THE EQUILIBRIUM OF PARAMETRICALLY PERTURBED SYSTEMS†

S. P. SOSNITSKII
Kiev
e-mail: sosn@imath.kiev.ua

(*Received* 27 *April* 2004)

Quasi-linear systems with many degrees of freedom are investigated for low dissipation and periodic perturbation using Lyapunov's second method. The periodic perturbation can be of small or large amplitude. Criteria of the asymptotic stability of the systems investigated are derived, which can be characterized as the sufficient conditions for parametric imperturbability of the latter when there is a weak dissipative background. The proposed approach enables limiting cases of periodic perturbation to be considered, when the corresponding frequency may approach both zero and infinity. Extensions to the case of non-periodic perturbations which vary very slowly or very rapidly with time are possible.

It is well known [1, 2], that in the case of periodic perturbation, low dissipation does not emerge as a barrier, opposing parametric build-up. Hence, it is of interest to establish the dissipation threshold, on reaching which parametric resonance does not imply instability of the equilibrium. Such a threshold can be established using Lyapunov's function. In particular, in the development of a previous investigation [3] it was possible to obtain a relation between the amplitudes of small dissipative and small perturbing periodic forces, guaranteeing asymptotic stability of equilibrium.

1. FORMULATION OF THE PROBLEM

Consider a non-autonomous system with n degrees of freedom

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = -\alpha D(\omega t, \mathbf{q})\dot{\mathbf{q}} \tag{1.1}$$

the Lagrangian of which is given by the expression

$$L(\omega t, \mathbf{q}, \dot{\mathbf{q}}) = L_2(\omega t, \mathbf{q}, \dot{\mathbf{q}}) + L_0(\omega t, \mathbf{q}) = \frac{1}{2}\dot{\mathbf{q}}^T A(\omega t, \mathbf{q})\dot{\mathbf{q}} + L_0(\omega t, \mathbf{q}) \tag{1.2}$$

$$L(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C^{1,2,2}_{t\mathbf{q}\dot{\mathbf{q}}}(R \times D_{\mathbf{q}} \times R^n_{\dot{\mathbf{q}}})$$

where $L(\omega t, \mathbf{q}, \dot{\mathbf{q}})$ depends $(2\pi/\omega)$-periodically on t $(\omega > 0)$. Moreover, we will assume that

$$\begin{gathered} A = A_0 + A^*(\omega t, \mathbf{q}), \quad A^T = A, \quad A^*(\omega t, \mathbf{0}) = 0 \\ L_0(\omega t, \mathbf{q}) = \frac{1}{2}\mathbf{q}^T B(\omega t)\mathbf{q} + \tilde{L}_0(\omega t, \mathbf{q}), \quad \tilde{L}_0(\omega t, \mathbf{q}) = o(\|\mathbf{q}\|^2) \\ B(\omega t) = B_0 + \beta B_1(\omega t), \quad B^T = B \\ D(\omega t, \mathbf{q}) = D_0 + D^*(\omega t, \mathbf{q}), \quad D^*(\omega t, \mathbf{0}) = 0, \quad D_0^T = D_0 \end{gathered} \tag{1.3}$$

†*Prikl. Mat. Mekh.* Vol. 69, No. 4, pp. 612–623, 2005.

0021-8928/$—see front matter.
doi: 10.1016/j.jappmathmech.2005.07.005

Here A_0, B_0 and D_0 are constant symmetrical matrices with positive eigenvalues and $A^*(\omega t, \mathbf{q})$, $B_1(\omega t)$, $D^*(\omega t, \mathbf{q})$ are $n \times n$ matrices, $(2\pi/\omega)$-periodic in t. The small parameters $\alpha > 0$ and $\beta > 0$ represent the degree of dissipation and parametric perturbation respectively.

Bearing expressions (1.3) in mind, we can rewrite Eqs (1.1) in the form

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + B(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}), \quad \|\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})\| = o(\|(\mathbf{q}, \dot{\mathbf{q}})\|) \tag{1.4}$$

As can be seen, the point $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ corresponds to an equilibrium position of this system. Henceforth, we will always mean by the word "equilibrium" this particular trivial solution of the system considered.

Together with (1.4) we will consider the truncated system

$$A_0\ddot{\mathbf{q}} + B_0\mathbf{q} = \mathbf{0} \tag{1.5}$$

which is obtain from system (1.4) when $\alpha = \beta = 0$ and the non-linear terms are dropped. Suppose $\omega_i > 0$ $(i = 1, 2, \dots, n)$ are the natural frequencies of system (1.5). As is well known [4] (see also [2]), satisfaction of the relation

$$\omega = (\omega_i \pm \omega_j)/k, \quad i, j = 1, 2, \dots, n, \quad k = 1, 2, \dots \tag{1.6}$$

which connects the frequency ω of the periodic perturbation and the natural frequencies ω_i of the truncated system (1.5) and so-called simple resonance, when $i = j$, and combination resonance in the opposite case, may imply instability of the equilibrium of parametrically perturbed systems.

Later it will important to clarify for which dissipation conditions the occurrence of instability of system (1.4), which arises as a consequence of resonance, can be prevented.

Henceforth it will be natural to assume that the parameter β is so small that the eigenvalues $b_i(t)$ of the matrix $B(\omega t) = B_0 + \beta B_1(\omega t)$ are positive, when $b_i(t) \geq \bar{b}_i$, $0 < \bar{b}_i = \text{const}$.

2. THE THEOREM OF ASYMPTOTIC STABILITY

Consider the equations

$$|D_0 - \lambda A_0| = 0, \quad |\partial B_1/\partial(\omega t) - \lambda B(\omega t)| = 0 \tag{2.1}$$

which correspond to the characteristic equations of corresponding pencils of quadratic forms [5]. The roots of Eqs (2.1) are the characteristic numbers of these pencils. Since the matrices A_0 and $B(\omega t)$ are positive-definite, each of Eqs (2.1) has n real roots λ_i $(i = 1, \dots, n)$.

Suppose further that λ^+ and λ^- are the greatest and least characteristic numbers of the first equation of (2.1), while the numbers μ^+ and μ^- correspond to $\sup(\mu_1(t), \dots, \mu_n(t))$ and $\inf(\mu_1(t), \dots, \mu_n(t))$, where $\mu_i(t)$ are the characteristic numbers of the second equation of (2.1).

Theorem 1. If the inequality

$$\alpha\lambda^- > \beta\mu^+\omega/2 \tag{2.2}$$

is satisfied, the equilibrium positive $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (1.4) is a asymptotically stable.

Proof. Representing system (1.4) in the form

$$\dot{\mathbf{q}} = A_0^{-1}\mathbf{p}, \quad \dot{\mathbf{p}} = -\alpha D_0\dot{\mathbf{q}} - B(\omega t)\mathbf{q} + \mathbf{F}^*(\omega t, \mathbf{q}, \mathbf{p}) \tag{2.3}$$

we consider the function

$$V = \frac{1}{2}\mathbf{p}^T A_0^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T B(\omega t)\mathbf{q} + \gamma\mathbf{q}\mathbf{p} + \frac{\alpha\gamma}{2}\mathbf{q}^T D_0\mathbf{q} \tag{2.4}$$

where γ is a positive constant, the choice of which is made below. The derivative of the function V with respect to t along the vector field, defined by system (2.3), can be conveniently written in the form

$$\frac{dV}{dt} = -[\dot{\mathbf{q}}^T(\alpha D_0)\dot{\mathbf{q}} - \gamma\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}}] + \left\{\mathbf{q}^T\left(\frac{\beta\omega}{2}\frac{\partial B_1}{\partial(\omega t)}\right)\mathbf{q} - \gamma\mathbf{q}^T B(\omega t)\mathbf{q}\right\} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$$

It can be seen that the right-hand side of this equation contains two regular pencils of quadratic forms, connecting the generalized velocities and generalized coordinates respectively. Now consider the characteristic equations of these pencils.

$$|\alpha D_0 - \gamma A_0| = 0, \quad \left|\frac{\beta\omega}{2}\frac{\partial B_1}{\partial(\omega t)} - \gamma B(\omega t)\right| = 0$$

Assuming $\gamma = \alpha\lambda$ in the first equation and $\gamma = \lambda\beta\omega/2$ in the second, we arrive at Eqs (2.1).

Starting from the extremal properties of the characteristic numbers of a regular pencil of quadratic forms [5], we obtain the inequalities

$$\lambda^- \le \frac{\dot{\mathbf{q}}^T D_0\dot{\mathbf{q}}}{\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}}} \le \lambda^+, \quad \mu^- \le \frac{\mathbf{q}^T\dfrac{\partial B_1}{\partial(\omega t)}\mathbf{q}}{\mathbf{q}^T B(\omega t)\mathbf{q}} \le \mu^+$$

on the basis of which we have

$$\lambda^-\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}} \le \dot{\mathbf{q}}^T D_0\dot{\mathbf{q}}, \quad \mathbf{q}^T\frac{\partial B_1}{\partial(\omega t)}\mathbf{q} \le \mu^+\mathbf{q}^T B(\omega t)\mathbf{q}$$

Moreover, from the condition for dV/dt to be negative-definite, we obtain the inequalities

$$\gamma\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}} < \alpha\dot{\mathbf{q}}^T D_0\dot{\mathbf{q}}, \quad \forall\|\dot{\mathbf{q}}\| \ne 0; \quad \frac{\beta\omega}{2}\mathbf{q}^T\frac{\partial B_1}{\partial(\omega t)}\mathbf{q} < \gamma\mathbf{q}^T B(\omega t)\mathbf{q}, \quad \forall\|\mathbf{q}\| \ne 0$$

Thus, taking inequality (2.2) into account and choosing the constant γ in accordance with the condition

$$\beta\mu^+\omega/2 < \gamma < \alpha\lambda^- \tag{2.5}$$

we arrive at the fact that the derivative dV/dt becomes negative-definite.

In order to prove the asymptotic stability of the equilibrium, we will show that the function V is positive-definite.

Consider the auxiliary system (the comparison system)

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + b^- E\mathbf{q} = \mathbf{0} \tag{2.6}$$

where $b^- = \min(\tilde{b}_1, \dots, \tilde{b}_n)$ and E is the identity matrix. Representing Eqs (2.6) in the form

$$\dot{\mathbf{q}} = A_0^{-1}\mathbf{p}, \quad \dot{\mathbf{p}} = -\alpha D_0\dot{\mathbf{q}} - b^- E\mathbf{q} \tag{2.7}$$

we consider the function

$$V^* = \frac{1}{2}\mathbf{p}^T A_0^{-1}\mathbf{p} + \frac{1}{2}b^-\mathbf{q}^T E\mathbf{q} + \gamma\mathbf{q}\mathbf{p} + \frac{\alpha\gamma}{2}\mathbf{q}^T D_0\mathbf{q} \tag{2.8}$$

By virtue of the system of equations (2.7) we have for its derivative

$$\frac{dV^*}{dt} = -\alpha\dot{\mathbf{q}}^T D_0\dot{\mathbf{q}} + \gamma\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}} - \gamma b^-\mathbf{q}^T E\mathbf{q}$$

and, therefore, when the second part of inequality (2.5) is satisfied, namely, $\gamma < \alpha\lambda^-$, the derivative dV^*/dt is negative-definite.

In system (2.6) the dissipation is complete. Consequently, where the second part of inequality (2.5) is satisfied the function V^* cannot be sign-variable, since this would contradict the asymptotic stability of the equilibrium of system (2.6). It is also cannot be degenerate, since this would contradict the fact dV^*/dt is sign-definite. Consequently, the function V^* is positive-definite.

Comparing the expressions for V and V^*, defined by Eqs (2.4) and (2.8) respectively, and bearing in mind the inequality

$$\mathbf{q}^T B(\omega t)\mathbf{q} \geq b^- \mathbf{q}^T E \mathbf{q}$$

we conclude that the function V is positive-definite. Hence, the equilibrium position of system (1.4) is asymptotically stable.

Remark 1. The proposed approach may turn out to be effective not only in the case of a periodic perturbation, but for any other oscillatory perturbation, which varies slowly with time, in particular, for systems of the form

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + [B_0 + \beta B_1(\varepsilon t)]\mathbf{q} = \mathbf{F}(\varepsilon t, \mathbf{q}, \dot{\mathbf{q}})$$

where ε is a small parameter.

Corollary 1. The equilibrium position of system (1.4) is asymptotically stable if one of the following conditions is satisfied:

(1) all the eigenvalues of the matrix

$$\alpha D_0 - (\beta\mu^+\omega/2)A_0 \tag{2.9}$$

are positive;

(2) the lower bound of the eigenvalues of the matrix

$$\alpha\lambda^- B(\omega t) - (\beta\omega/2)\partial B_1/\partial(\omega t) \tag{2.10}$$

is a positive number.

Proof. The correctness of the corollary follows from the fact that the limitations imposed on the eigenvalues of one of the matrices (2.9) and (2.10) exclude the satisfaction of the equation $\alpha\lambda^- = \beta\mu^+\omega/2$. The latter, if we take into account the procedure used when proving Theorem 1, only ensures that the eigenvalues of the above matrices are negative. Hence, when the conditions of the corollary are satisfied, inequality (2.2) is preserved and the condition of Theorem 1 is thereby satisfied.

Corollary 2. For small fixed parameters α and β, a threshold value of the frequency of the periodic perturbation

$$\omega_0 = \Omega, \quad \Omega = 2\frac{\alpha\lambda^-}{\beta\mu^+} \tag{2.11}$$

exist such that when $\omega < \omega_0$ the equilibrium position of system (1.4) is asymptotically stable.

Proof. The equation $\omega_0 = \Omega$ can be interpreted as the limiting relation for inequality (2.2). On the other hand, on the basis of the latter we have

$$\omega < \Omega \tag{2.12}$$

Since inequality (2.12) ensures that dV/dt is sign-definite, then according to relations (2.12) and (2.11) we can conclude that when $\omega < \omega_0$ the equilibrium position considered is asymptotically stable.

Corollary 3. For small fixed parameters α and β, the oscillations of a system with critical frequencies (1.6) and large values of k are quenched by dissipative forces. In particular, the limiting value k_0 of the number k, beginning from which the critical frequency (1.6) does not imply parametric build-up, is given by the inequality

$$k_0 - (\omega_i \pm \omega_j)/\Omega > 0; \quad \omega_i - \omega_j > 0$$

Proof. Assuming

$$\omega = (\omega_i \pm \omega_j)/k, \quad i, j = 1, 2, \ldots, n, \quad k = 1, 2, \ldots$$

in inequality (2.2), where $\omega_i - \omega_j > 0$, since $\omega > 0$ according to the initial propositions, we obtain the following lower limit for k

$$k > (\omega_i \pm \omega_j)/\Omega \tag{2.13}$$

Now evaluating the integer part of the right-hand side of inequality (2.13) and adding unity, we determine k_0.

Example 1. We will apply Theorem 1 to the problem of the oscillations of a pendulum in a periodically varying gravity field and a resisting medium [6, p. 28]. The equation of motion, apart from the notation, in these case has the form

$$\ddot{x} + \alpha\dot{x} + \gamma^2(1 + \beta\cos\omega t)x = o(\|(x, \dot{x})\|) \tag{2.14}$$

where α, β and γ are positive constants, where, as previously, we will assume that α and β are small quantities. By Theorem 1, the condition for asymptotic stability of the trivial solution $x = \dot{x} = 0$ has the form

$$\alpha > \frac{\beta\omega}{2}\max\left[-\frac{\sin\omega t}{1 + \beta\cos\omega t}\right] \tag{2.15}$$

Noting that the expression in square brackets on the right-hand side of inequality (2.15) takes an extremal value when $\beta + \cos\omega t = 0$, we arrive at the condition for asymptotic stability

$$\alpha > \frac{\beta\omega}{2}\max\frac{\pm\sqrt{1-\beta^2}}{1-\beta^2} = \frac{\omega}{2}\frac{\beta}{\sqrt{1-\beta^2}}$$

which is identical with the condition obtained previously by Starzhinskii [7].

Remark 2. As can be seen from Eq. (2.14) and the form of the coefficients occurring in it, the dimensions of the latter are not identical. This, however, has no effect on the correctness of the final result. Nevertheless, in more complex cases, to avoid misunderstandings, it is preferable to change to the dimensionless time $\tau = \lambda t$, where $\lambda = 1c^{-1}$ is the dimensional unity. As a result, all the coefficients become dimensionless while at the same time preserving their absolute value.

3. THE ASYMPTOTIC STABILITY OF THE EQUILIBRIUM FOR HIGH-FREQUENCY PERTURBATION

We will now consider a periodic perturbation with a high frequency ω, including the limiting case when $\omega \to \infty$. As will be shown below, when $\omega \to \infty$ the asymptotic stability of the equilibrium can be established for less restrictive assumptions regarding the periodic perturbation, in particular, its smallness. The latter fact enables us to consider the more general equations

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + B(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{3.1}$$

where

$$B(\omega t) \in C_t^0, \quad \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \in C_{t\mathbf{q}\dot{\mathbf{q}}}^{0,1,1}(R \times D_{\mathbf{q}\dot{\mathbf{q}}}), \quad \|\mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}})\| = o(\|(\mathbf{q}, \dot{\mathbf{q}})\|)$$

while the constant matrices A_0 and D_0, as before, are symmetrical with positive eigenvalues, and α is a small positive parameter.

We will further put

$$B^0 = \langle B\rangle = \frac{1}{T}\int_0^T B(\omega t)dt, \quad T = \frac{2\pi}{\omega}$$

$$B(\omega t) = B^0 + B - B^0 = B^0 + B_1(\omega t), \quad \langle B_1(\omega t)\rangle = 0$$

As can be seen, in this section, unlike the first two, the structure of the matrix $B(\omega t)$ is such that the periodic perturbation $B_1(\omega t)$ is not multiplied by a small parameter. Suppose $B_1^*(\omega t)$ is the original matrix $B_1(\omega t)$ such that $\langle B_1^*(\omega t)\rangle = 0$. Since the original matrix $B_1^*(\omega t)$ contains the quantity $1/\omega$ as a factor, it is convenient henceforth to represent it in the form $B_1^*(\omega t) = \tilde{B}_1(\omega t)/\omega$, where $\tilde{B}_1(\omega t)$, as before, is a matrix with zero mean.

Theorem 2. A threshold value of the frequency $\omega = \omega_0$ exists such that when $\omega > \omega_0$ the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (3.1) is asymptotically stable, if the following conditions are satisfied:

(1) the matrices A_0, D_0, $B^0 = \langle B\rangle$ are symmetrical and positive-definite, where the elements of the matrix A_0 are independent of ω, while the lower bounds of the eigenvalues of the matrices D_0 and B^0 satisfy the following inequalities respectively

$$d_0^- > d^*, \quad b^{0-} > b^*, \quad \forall\omega \in [\omega^*, \infty[\quad (0 < \omega^* = \text{const}) \tag{3.2}$$

where d^* and b^* are positive constants;

(2) $\lim B_1^* = 0$, $\lim B_1^{*T} A_0^{-1} D_0 = 0$, $\lim B_1^{*T} A_0^{-1} B = 0$ when $\omega \to \infty$.

Proof. We will represent Eqs (3.1) in the form

$$\dot{\mathbf{q}} = A_0^{-1}\mathbf{p}, \quad \dot{\mathbf{p}} = -\alpha D_0\dot{\mathbf{q}} - B(\omega t)\mathbf{q} + \mathbf{F}^*(\omega t, \mathbf{q}, \mathbf{p}) \tag{3.3}$$

Consider the function

$$V = V_0 + \frac{1}{\omega}\{\gamma\mathbf{q}^T\tilde{B}_1\mathbf{q} + \mathbf{q}^T\tilde{B}_1^T A_0^{-1}\mathbf{p}\} \tag{3.4}$$

where

$$V_0 = \frac{1}{2}\mathbf{p}^T A_0^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T B^0\mathbf{q} + \gamma\mathbf{q}\mathbf{p} + \frac{\alpha\gamma}{2}\mathbf{q}^T D_0\mathbf{q} \tag{3.5}$$

while the positive constant γ satisfies the inequality $\alpha d_0^- > \gamma a_0^+$, in which a_0^+ is the greatest eigenvalue of the matrix A_0. The derivative of the function V with respect to t along the vector field, defined by system (3.3), has the form

$$\frac{dV}{dt} = \frac{dV_0}{dt} + \frac{1}{\omega}\{\mathbf{q}^T[\gamma(\tilde{B}_1^T + \tilde{B}_1) - \alpha\tilde{B}_1^T A_0^{-1} D_0]\dot{\mathbf{q}} + \dot{\mathbf{q}}^T\tilde{B}_1^T\dot{\mathbf{q}} - \mathbf{q}^T\tilde{B}_1^T A_0^{-1} B\mathbf{q}\} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2) \tag{3.6}$$

where

$$\frac{dV_0}{dt} = -\alpha\dot{\mathbf{q}}^T D_0\dot{\mathbf{q}} + \gamma\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}} - \gamma\mathbf{q}^T B^0\mathbf{q} \tag{3.7}$$

As can be seen from expressions (3.6) and (3.7), at a sufficiently high frequency of the periodic perturbation ω and when conditions 1 and 2 of the theorem are satisfied, the derivative dV/dt is negative-definite, when the possibility of choosing the constant γ (which is independent of ω) is taken into account. The function V for sufficiently high ω, according to Eq. (3.4), is defined by expression (3.5) and is positive-definite. In order to show this, it is sufficient, following the proof of Theorem 1, to consider the comparison system

$$A_0\ddot{\mathbf{q}} + \alpha D_0\ddot{\mathbf{q}} + B^0\mathbf{q} = \mathbf{0}$$

Consequently, there is a complete basis for concluding that the equilibrium position in question is asymptotically stable when the conditions of the theorem are satisfied.

Remark 3. In the special case when the elements of the matrices B^0 and D_0 are independent of ω, we can drop the second part of condition 1, connected with inequalities (3.2).

Example 2. We will illustrate Theorem 2 on a system with one degree of freedom:

$$\ddot{x} + \alpha d_0 \dot{x} + (b_0 + \gamma\omega^\delta \cos\omega t)x = o(\|(x, \dot{x})\|)$$

where α, d_0, b_0 γ, δ are positive constants, and, as previously, α is a small parameter.

The satisfaction of condition 1 of Theorem 2 for the system considered is obvious, and condition 2 takes the form

$$\lim\omega^{\delta-1}[\gamma\sin\omega t] = 0, \quad \lim\omega^{\delta-1}[d_0\gamma\sin\omega t] = 0$$

$$\lim\omega^{\delta-1}[\gamma\sin\omega t](b_0 + \gamma\omega^\delta\cos\omega t) = 0 \text{ when } \omega\to\infty$$

Therefore, for a sufficiently large value of ω the equilibrium $x = \dot{x} = 0$ is asymptotically stable if $2\delta < 1$.

It follows from this example that, in the framework of Theorem 2, the form of the dependence of the matrices A_0, D_0 and $B(\omega t)$ on the parameter ω is important. As will be shown below (Example 5), this fact is due not only to the construction of the function V but also to the properties of the parametrically perturbed systems themselves. In this connection it is convenient to distinguish a class of systems for which conditions 1 and 2 of the theorem are always satisfied.

Definition. We will say that a periodic function (a vector function or a matrix function) $\Phi(\omega t) \in C_t^0$ only contains a parameter ω under the argument sign, if the parameter ω only occurs as a factor for t.

Functions of this class will be distinguished by a hat. In fact, if a periodic function belongs to this class: $\Phi(\omega t) = \hat{\Phi}(\omega t)$, this means that its Fourier coefficients are independent of ω. In the light of this definition, the following corollary of Theorem 2 holds.

Corollary. Suppose the matrices A_0, D_0, $B^0 = \langle B\rangle$ are symmetrical and positive-definite, where the elements of the matrices A_0 and D_0 are independent of ω, and $B(\omega t) = \hat{B}(\omega t)$. Then, a threshold value of the frequency $\omega = \omega_0$ exists, such that when $\omega > \omega_0$ the equilibrium position of system (3.1) is asymptotically stable.

Example 3. When considering Bolotin's problem of the dynamic stability of the plane form of the bending of an elastic beam [8], it was shown in [2] that a finite-dimensional linear model of the problem, which reduces to a system of the form

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + [B_0 + \beta\varphi(\omega t)B_1^*]\mathbf{q} = \mathbf{0} \tag{3.8}$$

may turn out to be useful. Here A_0, D_0 and B_0 are constant positive-definite diagonal matrices, not containing the parameter ω, $\varphi(\omega t)$ is a scalar periodic function, B_1^* is an asymmetrical constant matrix, also independent of ω, and β is a numerical parameter.

According to the corollary derived above, the equilibrium position of system (3.8) for a fairly large value of ω is asymptotically stable, irrespective of the value of the numerical parameter β, if the following conditions are satisfied

(1) $\varphi(\omega t) = \hat{\varphi}(\omega t)$, (2) $\langle\varphi(\omega t)\rangle = 0$.

4. EXTENSION TO THE CASE OF VARIABLE DISSIPATION

The proposed approach enables us to consider the move general equations

$$A_0\ddot{\mathbf{q}} + \alpha D(\omega t)\dot{\mathbf{q}} + B(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{4.1}$$

where, unlike Eqs (3.1), the matrix of the dissipative forces $D(\omega t)$ is variable, while the matrices A_0 and $B(\omega t)$ are the same as in Section 3.

Further, while retaining the notation used in Section 3, we will additionally put

$$D^0 = \langle D \rangle = \frac{1}{T}\int_0^T D(\omega t)dt, \quad T = \frac{2\pi}{\omega}$$

$$D = D^0 + D - D^0 = D^0 + D_1(\omega t), \quad \langle D_1(\omega t)\rangle = 0$$

Suppose $D_1^*(\omega t)$ is the primitive of the matrix $D_1(\omega t)$, where $\langle D_1^*(\omega t)\rangle = 0$, and, in the same way as above, the representation $D_1^*(\omega t) = \tilde{D}_1(\omega t)/\omega$, $\langle \tilde{D}_1(\omega t)\rangle = 0$ holds.

Theorem 3. Suppose the matrices $D^0 = \langle D(\omega t)\rangle$ and $B^0 = \langle B(\omega t)\rangle$ are symmetrical and, moreover, all the eigenvalues of the matrix D^0 are positive, while the elements of the matrices A_0, $D(\omega t)$ and $B(\omega t)$ respectively satisfy the limits

$$a_{0ij}^- \le a_{0ij} \le a_{0ij}^+, \quad d_{ij}^- \le d_{ij} \le d_{ij}^+, \quad b_{ij}^- \le b_{ij} \le b_{ij}^+, \quad \forall \omega \in [\omega^*, \infty[, \quad 0 < \omega^* = \text{const} \tag{4.2}$$

where the constants a_{0ij}^-, a_{0ij}^+, d_{ij}^-, d_{ij}^+, b_{ij}^-, b_{ij}^+ are independent of ω.

Then a threshold value of the frequency $\omega = \omega_0$ exists such that when $\omega > \omega_0$ the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (4.1) is asymptotically stable, if all the eigenvalues of the matrix $B^0 = \langle B(\omega t)\rangle$ are positive, and, conversely, unstable if they are all negative.

Proof. Representing Eqs (4.1) in the form

$$\dot{\mathbf{q}} = A_0^{-1}\mathbf{p}, \quad \dot{\mathbf{p}} = -\alpha D(\omega t)A_0^{-1}\mathbf{p} - B(\omega t)\mathbf{q} + \mathbf{F}^*(\omega t, \mathbf{q}, \mathbf{p}) \tag{4.3}$$

we consider the function

$$V = V^* + \frac{1}{\omega}\{\gamma \mathbf{q}^T\tilde{B}_1\mathbf{q} + \mathbf{q}^T(\tilde{B}_1^T + \alpha\gamma\tilde{D}_1)A_0^{-1}\mathbf{p} + \alpha\mathbf{p}^TA_0^{-1}\tilde{D}_1A_0^{-1}\mathbf{p}\} \tag{4.4}$$

The function V^* is defined by the right-hand side of (3.5) with D_0 replaced by D^0 in it, while the constant γ satisfies the inequality $\alpha d^{0-} > \gamma a_0^+$, d^{0-} is the least eigenvalue of the matrix D^0 and a_0^+ is the greatest eigenvalue of the matrix A_0.

The derivative of the function V with respect to t along the vector field, defined by system (4.3), has the form

$$\begin{aligned}\frac{dV}{dt} = \frac{dV^*}{dt} + \frac{1}{\omega}\{&\dot{\mathbf{q}}^T[\gamma(\tilde{B}_1^T + \tilde{B}_1) - \alpha(\tilde{D}_1^T + \tilde{D}_1)A_0^{-1}B]\mathbf{q} - \alpha\mathbf{q}^T(\tilde{B}_1^T + \alpha\gamma\tilde{D}_1)A_0^{-1}D\dot{\mathbf{q}} + \\ &+ \dot{\mathbf{q}}^T[\tilde{B}_1^T + \alpha\gamma\tilde{D}_1 - \alpha^2(\tilde{D}_1^T + \tilde{D}_1)A_0^{-1}D]\dot{\mathbf{q}} - \mathbf{q}^T(\tilde{B}_1^T + \alpha\gamma\tilde{D}_1)A_0^{-1}B\mathbf{q}\} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)\end{aligned} \tag{4.5}$$

The derivative dV^*/dt is defined by the right-hand side of Eq. (3.7) with D_0 replaced by D^0 in it.

If all the eigenvalues of the matrix B^0 are positive, then, as follows from expression (4.5), if we take into account the limits (4.2) and the choice of the constant γ, for sufficiently high values of ω the derivative dV/dt is negative-definite. In this case the sign of the function V itself is determined by the sign of the function V^*. The latter is positive-definite in the case considered. In order to convince ourselves of this, we follow the proof of Theorem 1, by considering the following equations as the comparison system

$$A_0\ddot{\mathbf{q}} + \alpha D^0\dot{\mathbf{q}} + B^0\mathbf{q} = \mathbf{0}$$

If all the eigenvalues of the matrix B^0 are negative, then, by choosing the number γ in accordance with the inequality $\alpha d^{0+} < \gamma a_0^-$, where d^{0+} is the greatest eigenvalue of the matrix D^0 and a_0^- is the least eigenvalue of the matrix A_0, for sufficiently large ω we can make the derivative dV/dt positive-definite.

Using the comparison system derived above, we will show that the function V in this case can take positive values.

Theorem 3 is proved.

The situation covered by Theorem 3, is similar to that considered in the corollary to Theorem 2. Unlike the corollary, here we have assumed that the Fourier coefficients of the periodic function depend on the parameter ω. However, the coefficients which depend on ω, in this case, are majorized by constant, which are independent of ω.

Example 4. We will illustrate Theorem 3 on the following system

$$\ddot{x} + \alpha[d_0 + e^{-\omega}\varphi_1(\omega t)]\dot{x} + \left[b_0 + \frac{\omega}{1+\omega}\varphi_2(\omega t)\right]x = o(\|(x, \dot{x})\|)$$

where α and d_0 are positive constants, and, as before, α is a small parameter. We will assume that

$$\varphi_i(\omega t) = \hat{\varphi}_i(\omega t), \quad i = 1, 2, \quad \langle\varphi_1(\omega t)\rangle = \varphi_1^0 > 0, \quad \langle\varphi_2(\omega t)\rangle = 0$$

Since the following inequalities hold

$$0 < e^{-\omega} \le e^{-\omega^*}, \quad \frac{\omega^*}{1+\omega^*} \le \frac{\omega}{1+\omega} < 1, \quad \forall\omega \in [\omega^*, \infty[\quad (0 < \omega^* = \text{const})$$

then, when $b_0 > 0$ and ω is sufficiently high, the equilibrium $x = \dot{x} = 0$ is asymptotically stable. Conversely, it is unstable if $b_0 < 0$.

Example 5. In connection with the example given above, it is of interest to return to the well-known problem of the stabilization of the upper position of a pendulum. The equation of motion in this case has the form

$$\ddot{x} + \alpha\dot{x} + (-b + a\omega^2\sin\omega t)x = o(\|(x, \dot{x})\|)$$

where α, a and b are positive constants. As before, α is a small parameter.

Unlike the situation considered above, here, for a sufficiently high value of ω, the equilibrium $x = \dot{x} = 0$ is asymptotically stable, although the average of the coefficient of x is negative. However, it is impossible not to note that in this problem the conditions of Theorem 3 are not satisfied; in particular, inequality (4.2) for the coefficient of x is not satisfied. Hence, inequality (4.2), which occurs in the conditions of Theorem 3, is important, and, depending on whether it is satisfied or not, the behaviour of the system is changed fundamentally.

Example 6. As one more example we will consider the system of equations related to vibrations of an elastic rod [2]

$$A_0\ddot{\mathbf{q}} + \alpha D_0\dot{\mathbf{q}} + [B_0 + \beta B_1(\omega t)]\mathbf{q} = \mathbf{0} \tag{4.6}$$

Here A_0, B_0 and D_0 are constant symmetrical positive-definite matrices, where the matrices A_0 and B_0 are diagonal, $B_1(\omega t) = \varphi(\omega t)B_1^*$, where $\varphi(\omega t)$ is a scalar periodic function, B_1^* is a constant symmetrical matrix, and β is a numerical parameter. The matrices A_0, B_0, D_0 and B_1^* do not contain the parameter ω.

We will further assume, for simplicity, that $\varphi(\omega t) = \hat{\varphi}(\omega t)$. Then, by Theorem 3, a threshold value of the frequency $\omega = \omega_0$ exists, such that when $\omega > \omega_0$ the equilibrium position of system (4.6) is asymptotically stable, if all the eigenvalues of the matrix are positive and, conversely, unstable, if they are all negative.

Remark 4. The proposed approach may turn out to be useful not only in the case of periodic perturbation, but also for any other rapidly oscillating perturbation, in particular, for systems of the form

$$A_0\ddot{\mathbf{q}} + \alpha D(\lambda t)\dot{\mathbf{q}} + B(\lambda t)\mathbf{q} = \mathbf{F}(\lambda t, \mathbf{q}, \dot{\mathbf{q}})$$

where λ is a large parameter. Of course, in this case it is necessary for an average of the matrices $D(\lambda t)$ and $B(\lambda t)$ to exist.

5. AN ANALOGUE OF THEOREM 1 UNDER CONDITIONS OF VARIABLE LOW DISSIPATION

It is of practical interests to obtain an analogue of Theorem 1 in the situation when the low dissipation is variable, and the corresponding equations take the form

$$A_0\ddot{\mathbf{q}} + \alpha D(\omega t)\dot{\mathbf{q}} + B(\omega t)\mathbf{q} = \mathbf{F}(\omega t, \mathbf{q}, \dot{\mathbf{q}}) \tag{5.1}$$

where

$$D(\omega t) = [D_0 + \beta D_1(\omega t)], \quad D_1(\omega t)^T = D_1(\omega t)$$

while the matrices A_0, D_0 and $B(\omega t)$ are the same in Section 1.

It is natural to assume further that the small parameter β is so small that the eigenvalues $d_i(t)$ and $b_i(t)$ of the matrices $D(\omega t)$ and $B(\omega t)$ respectively, are positive, and

$$d_i(t) \geq \tilde{d}_i, \quad b_i(t) \geq \tilde{b}_i, \quad 0 < \tilde{d}_i = \text{const}, \quad 0 < \tilde{b}_i = \text{const}$$

Moreover, we will assume that the eigenvalues of the matrix

$$B^*(\omega t) = B(\omega t) - \frac{\alpha\beta\omega}{2}\frac{\partial D_1}{\partial(\omega t)}$$

are also positive, and, in particular, their lower bound exceeds a certain positive constant.

Consider the equations

$$|D(\omega t) - \lambda A_0| = 0, \quad \left|\frac{\partial B_1}{\partial(\omega t)} - \lambda B^*(\omega t)\right| = 0 \tag{5.2}$$

Since the matrices A_0 and $B^*(\omega t)$ are positive-definite, each of the equations (5.2) has n real roots $\lambda_1 \ldots , \lambda_n$.

Suppose further that the numbers λ^+ and λ^- correspond to the upper and lower bounds of the characteristic numbers of the first equation of (5.2), while the numbers μ^+ and μ^- correspond to the upper and lower bounds of the characteristic numbers of the second equation.

Theorem 4. If the following inequality holds

$$\alpha\lambda^- > \beta\mu^+\omega/2 \tag{5.3}$$

the equilibrium position $\mathbf{q} = \dot{\mathbf{q}} = \mathbf{0}$ of system (5.1) is asymptotically stable.

Proof. Representing system (5.1) in the form

$$\dot{\mathbf{q}} = A_0^{-1}\mathbf{p}, \quad \dot{\mathbf{p}} = -\alpha D(\omega t)\dot{\mathbf{q}} - B(\omega t)\mathbf{q} + \mathbf{F}^*(\omega t, \mathbf{q}, \mathbf{p}) \tag{5.4}$$

we consider the function

$$V = \frac{1}{2}\mathbf{p}^T A_0^{-1}\mathbf{p} + \frac{1}{2}\mathbf{q}^T B(\omega t)\mathbf{q} + \gamma\mathbf{q}\mathbf{p} + \frac{\alpha\gamma}{2}\mathbf{q}^T D(\omega t)\mathbf{q} \tag{5.5}$$

where γ is a positive constant, which will chosen below. We will write the derivative of the function V with respect to the vector field, defined by system (5.4), in the form

$$\frac{dV}{dt} = -\{\alpha\dot{\mathbf{q}}^T D(\omega t)\dot{\mathbf{q}} - \gamma\dot{\mathbf{q}}^T A_0\dot{\mathbf{q}}\} + \left\{\mathbf{q}^T\left(\frac{\beta\omega}{2}\frac{\partial B_1}{\partial(\omega t)}\right)\mathbf{q} - \gamma\mathbf{q}^T B^*(\omega t)\mathbf{q}\right\} + o(\|(\mathbf{q}, \dot{\mathbf{q}})\|^2)$$

Proceeding as in the proof of Theorem 1, we reach the conclusion that the choice of the constant γ in accordance with the condition

$$\beta\mu^+\omega/2 < \gamma < \alpha\lambda^-$$

ensures that the derivative dV/dt is negative-definite.

In order to prove that the function V is positive-definite, we will choose the system of equations in the form

$$A_0\ddot{\mathbf{q}} + \alpha d^- E\dot{\mathbf{q}} + b^- E\mathbf{q} = \mathbf{0}$$

where $d^- = \min(\tilde{d}_1, \dots, \tilde{d}_n)$, $b^- = \min(\tilde{b}_1, \dots, \tilde{b}_n)$.

By repeating the proof of Theorem 1 almost word for word, we can conclude that Theorem 4 is true.

Corollary. The equilibrium position of system (5.1) is asymptotically stable if the lower bound of the eigenvalues of one of the matrices

$$\alpha D(\omega t) - \frac{1}{2}\beta\mu^+\omega A_0 \quad \text{or} \quad \alpha\lambda^- B^*(\omega t) - \frac{1}{2}\beta\omega\frac{\partial B_1}{\partial(\omega t)}$$

is positive.

Analogues of Corollaries 2 and 3 of Theorem 1 can be formulated and proved in the same way as in Section 2. The only difference is the fact that λ^- and μ^+ are now defined by Eqs (5.2).

In conclusion, we note that the sufficient conditions for the auxiliary V-functions to be Lyapunov functions are in fact reflected in the formulations of Theorems 1–4 [9]. In specific situations which, incidentally, are partly mentioned in the Remarks to Sections 2 and 4, the possibility of using the V-functions themselves may turn out to be wider than the conditions for which the theorems were derived.

REFERENCES

1. SCHMIDT, G., *Parametererregte Schwingungen*. Deutsch. Verl. Wissenschaft, Berlin, 1975.
2. MAILYBAYEV, A. A. and SEYIRANIAN, A. P., Parametric resonance in systems with low dissipation. *Prikl. Mat. Mekh.*, 2001, **65**, 5, 779–792.
3. SOSNITS'SKII, S. P., The asymptotic stability of the equilibrium of parametrically perturbed systems. In *Problems of Mechanics and its Applications*. Inst. Mat. NAN Ukraini, Kiev, 2002, **44**, 238–248.
4. YAAKUBOVICH, V. A. and STARZHINSKII, V. M., *Linear Differential Equations with Periodic Coefficients and their Applications*. Nauka, Moscow, 1972.
5. GANTMAKHER, F. R., *Matrix Theory*. Nauka, Moscow, 1967.
6. ANDRONOV, A. A., *Collected papers*. Izd. Akad. Nauk SSSR, Moscow, 1956.
7. STARZHINSKII, V. M., The sufficient conditions for the stability of a mechanical system with one degree of freedom. *Prikl. Mat. Mekh.*, 1952, **16**, 3, 269–374.
8. BOLOTIN, V. V. (Ed.), *Vibrations in Technology*, Vol. 1, *Oscillations of Linear Systems*. Mashinostroyeniye, Moscow, 1999.
9. LYAPUNOV, A. M., *The General Problem of the Stability of Motion*. Gostekhizdat, Moscow; Leningrad, 1950.

Translated by R.C.G.